Finding the center of a circle by constructions using a straightedge only-investigation of cases

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Abstract

Throughout the entire development period of mathematics, geometric constructions have been examined by renowned mathematicians, each of whom made his own contribution to the understanding of the possibilities of constructions. Over the years, other branches were added to geometric constructions, such as: construction using a straightedge only, construction using a compass only and construction using a graded ruler.

In the paper we present original, previously unknown tasks of finding the center of a circle by using a straightedge only. The tasks were given as part of a course in "geometric constructions" given to students in a post-primary route of teachers' training in mathematics.

In each task we present a description of the construction based on an investigation carried out by the pre-service teachers using dynamic software (Geogebra) so as to bring the students to use computer tools, and to harness the capabilities of these tools to improve the teaching and learning processes. The surprising aspect of the tasks is that although only the use of the straightedge was allowed, they can still be accomplished.

1. Introduction

The subject of geometric constructions using a straightedge and a compass has been the focus of attention of geometers over many generations. The Danish mathematician Mohr (from the 17th century) and the Italian mathematician Masceroni (from the 18th century) have stated that any construction that can be carried out using a straightedge and compass can also be carried out using a compass only [1].

Later the Swiss mathematician Jakob Steiner proved that many constructions can be carried out using a straightedge only, if in addition to the straightedge, a certain geometric form is drawn in the plane. The famous Poncelet-Steiner theorem states that any construction that can be carried out using a straightedge and a compass can also be carried out using a straightedge only if a circle and its center are drawn in the construction plane; in other words – a straightedge and a fixed circle with a given center are equivalent to a straightedge and a compass [1,2].

Thus it is relevant to find such constructions when we have some additional geometric forms in the plane. Some of these constructions will be considered in this article.

The results presented in this paper were obtained from joint investigative work of lecturers and students in the post-primary route of teacher's training in mathematics of the academic college "Shaanan" in Haifa (Israel). It is important to note that some of the students are active teachers who studied for a qualification certificate in teaching mathematics.

Before carrying out the investigative work, the students studies basic courses in the following subjects: algebra, plane geometry, trigonometry, analytic geometry, differential and integral calculus and other courses in mathematics, after which the students passed a course in the integration of computerized technology in the teaching of mathematics.

The investigative work was carried out as part of the courses in the history of mathematics and the seminar for mathematical enrichment. In the courses we presented the subject of geometric constructions from ancient times until today, when constructions can be carried out using computerized software. The role of the lecturers was mainly guidance and instruction. The students were required to analyze the data required to find the center of the circle and to prove the correctness of the construction. Subsequently, they were required to find the center of the circle using a straightedge only after appropriate training of carrying out the basic constructions using the applets of the dynamic software (GeoGebra).

The subject was presented in a comprehensive manner starting from the geometric constructions in the ancient world and up to modern-world constructions, where some constructions are carried out using dynamic geometric software.

Research in education seeks ways to improve the quality of teaching and learning and it is therefore also focused on integration of technology in teaching. The technological tool allows one to construct and to represent mathematical objects in a dynamic manner while providing the user with feedback during the solution of problems [3,4]. Dynamic geometry environment (DGE) software serves as an intermediary tool used to bridge the gap between the physical model and symbolic proof[5]. Learning that includes use of a dynamic software allows the students to discover mathematical phenomena, mathematical models, different representations and connections between graphical descriptions while referring to mathematical concepts [6]. By learning using a technological tool students can describe mathematical concepts and relations better when compared to teaching that does not include a technological tool. The students showed greater understanding of these concepts and they are given access to high-level mathematical ideas [7].

Special attention was given to constructions in which limitations were imposed on the drawing tools, focusing on constructions using a straightedge only.

Since the students, as well as quite a large part of the teachers of mathematics are not familiar with the applications of Steiner's theorem for trapezoids (which allows the bases of a given trapezoid to be bisected) and more advanced geometric constructions using a straightedge only,
at the first stage the students were given four known basic constructions:

1. Construction 1 – finding the midpoints of parallel segments.

2. Construction 2 – constructing a parallel line using a straightedge only.

3. Construction 3 – constructing a perpendicular to the diameter in a circle.

4. Construction 4 – constructing a tangent to a circle at a point on the circle

Use will be made of these constructions during the execution of the tasks presented below, with the appropriate reference being made at each stage.

These constructions are provided as “tools” to the students to allow them to deal with a challenging construction task in this topic, which were not published in the professional literature, and which were prepared by the authors of the paper.

During the second stage three different tasks of finding the center of a circle using a straightedge only were given for the following cases:

a. Intersecting circles, including the cases where the circle is shifted to a position in which there is tangency from the inside and tangency from the outside.
b. Two disjoint circles, in each of which the diameter is given.
c. Disjoint circle and regular polygon.

During the third stage the students were given a triangle that was inscribed in a circle and they were required to add to it data such as medians, angle bisectors, altitudes, and to investigate for which data can some construction be used in order to find the center of the circle, **so that at any stage they would be able to specify the property or the theorem on which the construction was based.**

The students’ activity produced unfamiliar possibilities of construction.

2. Stage A

Presentation of four known constructions [2,8,9] carried out by a straightedge only and practicing construction using the applets of dynamic geometric software:

Construction 1 – finding the midpoints of parallel segments.

Given two parallel segments AB||CD, find their midpoints.

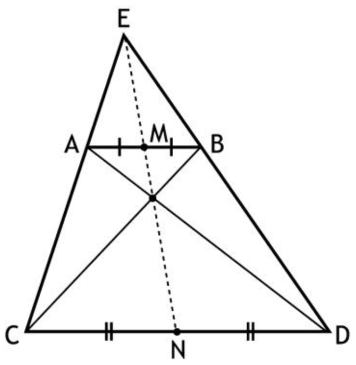

Figure 1

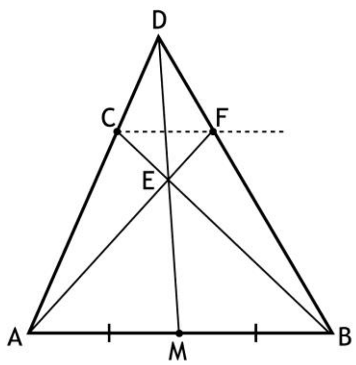

Figure 2

From Steiner’s theorem the straight line that connects the intersection of the diagonals of the trapezoid with the point of intersection of the continuation of the sides – bisects the bases of the trapezoid (Figure 1).

Construction 2 – constructing a parallel line using a straightedge only.

Given the segment AB and its midpoint M, Construct a straight line parallel to AB, so that it passes through any point C (Figure 2).

The full description of the construction using a straightedge only appears in [2, pp.51-52].

Construction 3 – constructing a perpendicular to the diameter in a circle (reference [8] gives a full description of the construction with its proof)

Given is a circle in which the diameter AB is marked, draw a perpendicular to the given diameter from any point C (Figure 3).

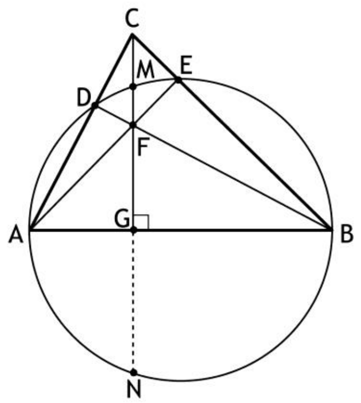

Figure 3

It is clear that the perpendicular CF intersects the circle at two points M and N, such that the point G is the middle of the segment MN. Use shall be made of this property in the tasks presented below.

Construction 4 – constructing a tangent to a circle at a point on the circle (Reference [2], pp.46-47, gives the method of construction of the tangent with a mathematical proof that uses tools of descriptive geometry. Reference [9] gives a proof that combines tools from trigonometry, analytic geometry and Euclidean geometry).

Given is a circle on which the point A is marked. Draw a tangent to the circle at this point using a straightedge only (Figure 4).

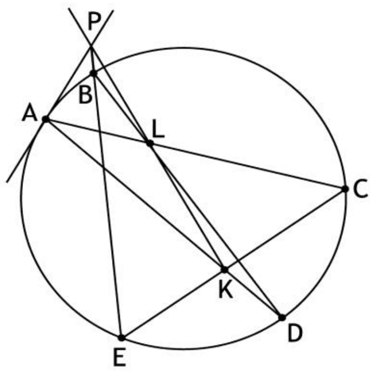

Figure 4

In order to practice each of the four constructions listed above, a Geogebra applet was prepared for each of them, which can be accessed using the following links:

Link 1- Finding the midpoints of parallel segments.

https://tube.geogebra.org/student/m351945

Link 2 - Constructing a parallel line using a straightedge only.
http://www.geogebratube.org/student/m121878

Link 3 - Constructing a perpendicular to the diameter in a circle.

http://tube.geogebra.org/student/m352489

Link 4 - Constructing a tangent to a circle at a point on the circle.

http://tube.geogebra.org/student/m352507

The ability to carry out these constructions, in addition to the ability to implement theorems in geometry constitute the “toolbox” for the execution of the subsequent tasks.

3.Stage B

Finding the center of a circle in three given tasks.

Task 1 – finding the center of a circle – intersecting circles

Given are two circles with different radii, which intersect at the points A and B. Find the centers of the circles using a straightedge only [2].

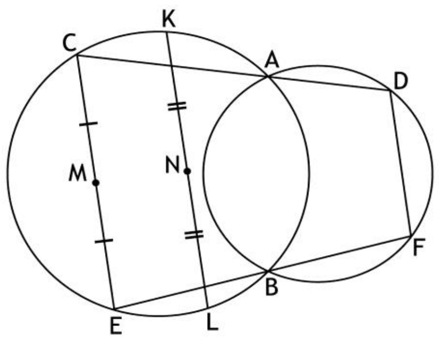

Figure 5a

Step A

In the left circle (Figure 5a), construct a chord AC, whose continuation is the chord AD in the right circle. In the left circle construct a chord EB, whose continuation is a chord in the second circle. The segments CE and DF are parallel.

Indeed, the quadrilateral ACEB is inscribed in a circle, therefore the sum of the angles CEB and CAB is 180°. In addition, quadrilateral ADFB is also inscribed in a circle, therefore, in this quadrilateral the sum of the angles BAD and BFD is also 180°. Since the angle CAB supplements the angle DAB to 180°, we obtain that the angle BFD is equal to the angle CAB, and therefore the sum of the angles BFD and CEB is 180°. It follows that CE and DF are parallel, i.e., CDFE is a trapezoid.

From Construction 1 one can find the midpoint M of the chord CE. On the arc CA we choose some point K, and using the Construction 2 we construct the chord KL that is parallel to the chord CE.

Using Construction 1 again, we find the point N – the midpoint of the chord KL. A diameter of the left circle lies on the segment MN and its continuation.

Step B

We construct two other chords C'D' and E'F' that pass through the point of intersection of the circles A and B respectively.

As in step A, we construct the midpoint M' of the chord C'E', we construct a chord K'L' parallel to C'E', and find its midpoint N'. On the segment M'N' and its continuations lies another diameter of the left circle. The point of intersection of the straight line that passes through the points M and N with the line that passes through the points M' and N' is the center of the left circle.

In the same manner, using the left circle, we find the center of the right circle.

Using the Geogebra software one can move one of the circles (closer or farther apart), with the points C and E remaining fixed and the chords passing through the points of intersection of the circles. After moving the circle there are several options:

1. The chords CF and ED do not intersect and a trapezoid is formed.
2. The chords CF and ED intersect at a point on the circle (F and D coincide) and a trapezoid is not formed.
3. The chords CF and ED intersect inside the circle and a trapezoid is formed.
4. The circles are tangent to each other from the inside or from the outside (A and B coincide). The chords CF and ED intersect at the point of tangency and a trapezoid is formed.

In the cases 1, 3 we carry out the same construction in order to obtain the center of the circle.

In the case 2, one first has to construct the tangent to the circle at the point at which the chords CF and ED intersect on the circle (Construction 4). The tangent to the circle at the point of intersection of the chords is parallel to the chord CE. We now choose two points on it and a trapezoid is formed.

In the case 4 we construct the inner tangent that is common to the two circles (Construction 4) and choose two points on it so that a trapezoid is formed.

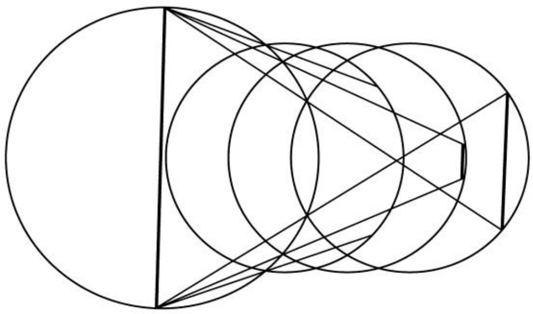

Figure 5b

An illustration of states 1-3 is given in Figure 5b, where the left circle remains in its place and the right circle is dragged to the right.

Dynamic viewing of the movement of the circles (closer or farther apart) can be carried out by using the following link:

Link 5 - Finding the center of a circle – intersecting circles.

http://www.geogebratube.org/student/m121881

It is possible to change the size of the right circle, by dragging the point R. When the pointer is on the right circle's perimeter, it is possible to drag this circle and see all the situations as shown in Figure 5b.

Task 2 – finding the center of the circle – Circles with a diameter

Given are two disjoint circles, where in each a diameter is drawn as shown in Figure 6. The diameterAB is given in the left circle, and the diameter CD is given in the right circle.

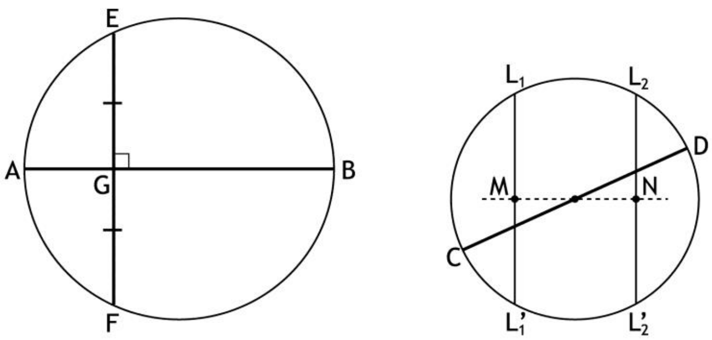

Figure 6

Description of the construction

In accordance with Construction 3, we construct at some point a perpendicular to the diameter AB of the left circle, which intersects the circle at the points EF. The point of intersection G with the diameter is the midpoint of the chord EF.

We choose some points L_1 and L_2 on the arc $\overset{\frown}{CD}$ of the right circle. Using the chord EF and its midpoint G, we construct two parallel lines $L_1L'_1$ and $L_2L'_2$, which are parallel to this chord using Construction 2. In accordance with Construction 1, we construct the midpoints M and N of the chords $L_1L'_1$ and $L_2L'_2$, respectively.

A diameter of the right circle (as a straight line connecting the middles of parallel chords) lies on the segment MN and on its continuation, therefore the point of intersection of the straight line connecting the points M and N with the diameter CD is the center of the right circle.

In a similar manner, using the right circle one finds the center of the left circle.

Task 3 – Disjoint circle and regular polygon

Given is a circle, the location of whose center has to be found using a straightedge only, and a regular polygon that is located near it (having five sides or more).

For example, Figure 7a shows a regular polygon with 9 sides.

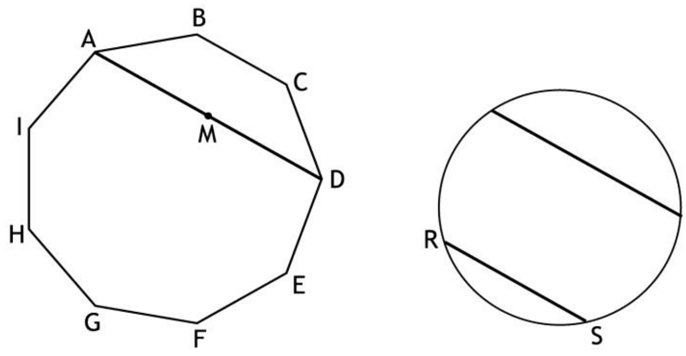

Figure 7a

<u>Description of the construction</u>

When connecting any vertex with the third vertex in sequence after it, an isosceles trapezoid is obtained. The proof of this is based on the fact that a regular polygon can be inscribed in a circle, and due to the equal sides (chords of the circle), an isosceles trapezoid is formed. For example, the quadrilateral ABCD is an isosceles trapezoid. Using Construction 1 (Steiner's theorem), one can find the midpoint M of the chord AD.

In the same manner we select four other vertices in sequence and connect the midpoint of another diagonal in the polygon. Using the two diagonals and their midpoint, through Construction 2 one can construct two pairs of parallel chords in the given circle. For example, the parallel chords RS and TQ were constructed using the diagonal AD of the polygon and its midpoint M. Using Construction 1, we find the midpoint of each pair of parallel chords. A diameter of the circle lies on each of the straight lines passing through the midpoints of a pair of parallel chords, and therefore their point of intersection is the center of the desired circle.

<u>Follow-up task</u>

Given are two disjoint circles. The diameter AB is given in the left circle, where its continuation does not pass through the center of the right circle. Using a straightedge only, we wish to find the location of the centers of the circles that are not given (Figure 7b).

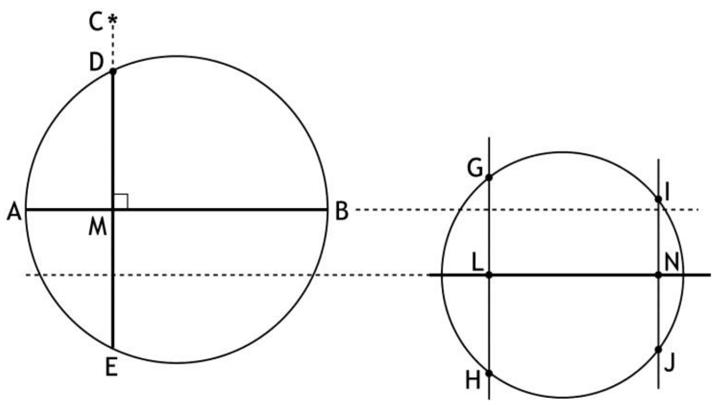

Figure 7b

Description of the construction

From some point C we drop a perpendicular to the diameter AB. This perpendicular intersects the diameter at the point M and the circle at the points D and E. The point M is the midpoint of the chord DE.

Using the segment DE and its midpoint we construct according to construction 2 two parallel lines that cut two chords GH and IJ off the right circle. We construct the midpoints L and N of these chords. A diameter of the right circle lies on the straight line LN, and we also have AB||LN. Using the parallel lines and their continuations, we obtain in each circle another pair of parallel chords, and another diameter can be constructed in each circle. The intersections of the diameters in each circle give their centers.

4.Stage C

Investigative tasks for constructing the center of the circle, as carried out by the students.

As specified in the introduction, in all of these tasks some triangle that is inscribed in the circle is given (aside for section 4 – the first section in these tasks, where an isosceles triangle is given), and one has to determine which data (medians, angle bisectors and altitudes) allow one to find the center of the circle using a straightedge only. In each construction the students were required to describe the process and to justify it.

Task 4a – Finding the center of a circle circumscribing an isosceles triangle

Given is an isosceles triangle ΔABC ($AB = AC$) inscribed in a circle, and the bisector BD of the base angle $\angle ABC$ is drawn, as shown in Figure 8.

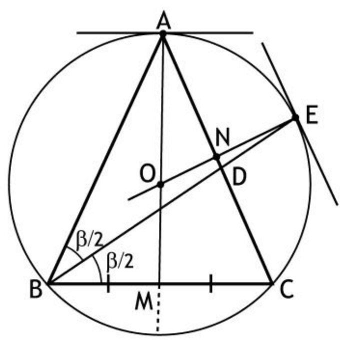

Figure 8

Description of the construction

We extend the angle bisector BD up to the point of its intersection with the circle (the point E).

By means of Construction 4 we construct tangents to the circle at the points E and A, as shown in Figure 8.

The tangent at point A is parallel to the base BC and the tangent at point E is parallel to the side AC. The proof of this is based on the angle between the tangent to a chord and the inscribed angles resting on equal arcs, and the resulting equal alternate angles between the tangent and the side of the triangle.

Using Construction 1, we construct the point M, the midpoint of the base BC, and since the triangle ΔABC is an isosceles triangle, a diameter of the circle lies on the segment AM and its continuation.

Since the tangent at point E is parallel to the side AC, using Construction 1 one can find the point N, which is the midpoint of the side AC.

Another diameter of the circle lies on the straight line NE, therefore the point of intersection of the straight line that passes through the points N and E and the straight line that passes through the points A and M is the required center of the circle (the point of intersection of two diameters - the point O in Figure 8).

<u>Task 4b – Finding the center of a circle circumscribing an arbitrary triangle in which two angle bisectors are given</u>

Given is some triangle ΔABC that is inscribed in a circle, where two angle bisectors are given as shown in Figure 9.

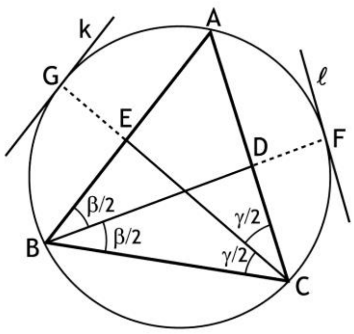

Figure 9

This task is a continuation of the previous task and it is based on the same method of construction.

<u>Description of the construction</u>

We extend the angle bisectors BD and CE up to their points of intersection with the circle, F and G respectively.

As in the previous task, would construct tangents *l* and *k* to the circle at the points F and G, respectively. These tangents are parallel to the sides of the triangle; *l*||AC and *k*||AB. The subsequent process is similar to the previous task (see also [10]).

Task 4c – Finding the center of a circle circumscribing any triangle in which two medians are given

Given is some triangle ΔABC that is inscribed in a circle, where two medians are given as shown in Figure 10.

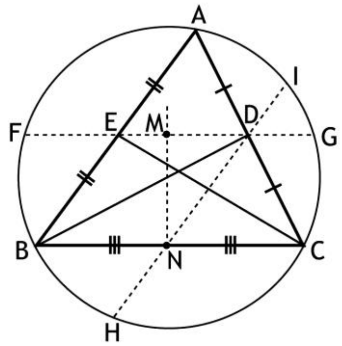

Figure 10

Description of the construction

The segment ED is a midline in the triangle, its continuations intersect the circle at the points G and F. The chords FG and BC are parallel to each other and therefore, using Construction 1, we find the location of the points M and N, which are the midpoints of the chords FG and BC, respectively.

A diameter of the circle lies on the straight line connecting the points M and N. The segment ND is also a midline in the triangle, and in the same manner we construct the point P – the midpoint of the chord HI that passes through the points N and D (the point P is not visible in figure in order not to overload him with marks). Another diameter of the circle lies on the straight line that connects the points P and E. The point of intersection of the straight line that passes through the points M and N and the straight line that passes through the points P and E is the center of the circle.

Task 4d – Finding the center of a circle circumscribing an arbitrary triangle in which a median and an altitude passing through the same vertex are given

Given is some triangle ΔABC that is inscribed in a circle, an altitude and a median issuing from the same vertex A are given in the triangle as shown in Figure 11.

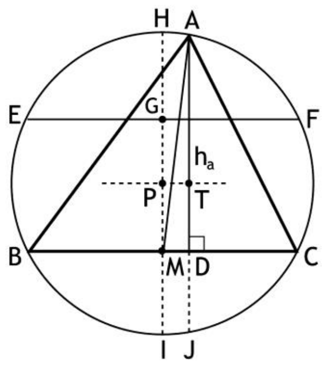

Figure 11

Using Construction 2 one can construct at any location in the circle a parallel line to the base BC that cuts a chord EF from the circle.

Using Construction 1 one can find the point G, which is the midpoint of the chord EF.

A diameter of the circle lies on the straight line connecting the points G and M.

The continuations of the segment MG intersect the circle at the points H and I. The continuation of the altitude AD intersects the circle at the point J. The straight lines HI and AJ are parallel to each other.

Using Construction 1 one can find the points P and T, which are the midpoints of the parallel chords HI and AJ, respectively.

The point P is the center of the circle.

<u>Task 4e – Finding the center of a circle circumscribing an arbitrary triangle in which a median and an angle bisector passing through different vertices are given</u>

Given is some triangle ΔABC that is inscribed in a circle, an angle bisector and a median issuing from the different vertexes of the base are given in the triangle as shown in Figure 12.

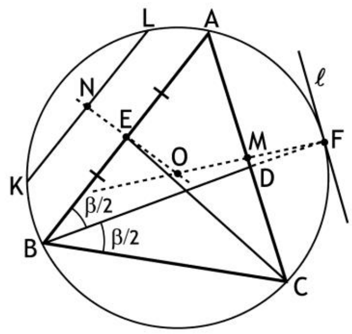

Figure 12

BD is an angle bisector whose continuation intersects the circle at the point F. CE is a median.

<u>Description of the construction</u>

At the point F we construct a tangent *l* to the circle in accordance with Construction 4. This tangent is parallel to the side of the triangle AC (*l*||AC) as explained in task 4a.

Using Construction 1 we find the point M – the midpoint of the side AC. A diameter of the circle lies on the straight line connecting the points F and M.

Using Construction 2 we construct a parallel to the side AB which intersects the circle at the points K and L. From Construction 1 we find the point N which is the midpoint of the chord KL. A diameter of the circle lies on the straight line connecting the points E and N. The straight line connecting the points M and F and the straight line connecting the points E and N intersect at the point O, which is the center of the circle.

<u>Task 4f – Finding the center of a circle circumscribing an arbitrary triangle in which an angle bisector and an altitude passing through the same point are given</u>

Given is some triangle ΔABC that is inscribed in a circle, an altitude and an angle bisector issuing from the same vertex are given in the triangle as shown in Figure 13.

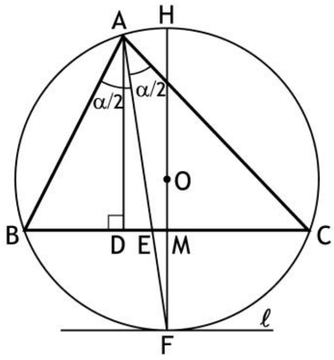

Figure 13

We extend the angle bisector AE up to the point of intersection with the circle – the point F. At this point we construct a tangent *l* to the circle. The tangent *l* is parallel to the base BC. The explanation for this is that in any triangle that is inscribed in a circle, the angle bisector from any vertex on the triangle and the midperpendicular to the side opposite that vertex intersect at a point that lies on the arc of the circle (due to equal arcs). Since a diameter rests on a midperpendicular to a chord in a circle, the tangent to the circle at the point of intersection of the bisector issuing from any vertex and the midperpendicular to the side opposite the vertex is parallel to the side on which the midperpendicular was constructed (equal angle between the radius and the tangent). By using Construction 1 we find the point M – the midpoint of the base.

The continuation of the straight line FM intersects the circle at the point H. The chord FH is perpendicular to the base BC and is therefore a diameter of the circle. Using the segment AD that is parallel to FH (AD||FH) and using Construction 1, we find the midpoint of the diameter FH, which is the center of the circle O.

In addition to these cases that were found by the students, which allow one to find, by construction using a straightedge only, the location of the center of a circle that inscribes a triangle, two students found that when an equilateral triangle is given, that is inscribed in a circle, its center can be found using constructions 1 and 2.

Summary

An investigative task in the presented which combines use of a straightedge only for finding the location of the center of a circle given additional geometric shapes with certain lines drawn in them, and demonstration of the method of finding the location of the center of a circle using dynamic geometric software.

The basic constructions presented at the first stage, the demonstration of the three tasks carried out during the second stage and the third stage in which tasks were presented whose solution were achieved by pre-service teachers of mathematics were set apart.

The investigative task constitutes the continuity of historic mathematical constructions and modern mathematical constructions, where some constructions are carried out using computer technology.

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